Practice in AIME Geometry

Zilu Pan and Tristan Pollner

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1 Introduction

Skill in geometry is essential to success on the AIME and contest math in general. Fortunately, getting better at geometry (whether computional or proof-based) is really only a question of practice as there is little prerequisite material. Today we will quickly cover basic theorems and strategies and then focus on solving AIME problems. In later sessions we will focus more on beginning olympiad (i.e. proof-based) geometry.

2 The Basics

In roughly decreasing order of importance.

1. Similar Triangles: Two similar triangles have congruent angles and proportional side lengths. Finding similar triangles is essential in Mathcounts, AMC, AIME, and Olympiads - always be on the lookout. Prove two triangles are similar using SSS, SAS, or AA.

2. Power of a Point: Let P be a point and ω a circle in the plane. If one line through P intersects ω at A and B, and another line intersects ω at C and D, $(PA)(PB) = (PC)(PD)$. (Note : This actually is just similar triangles - make sure you see why.)

3. Length/Area Relations: If K is the area of $\triangle ABC$, then

$$K = \frac{1}{2}bh = \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}$$

4. Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

   Think of this if you see angles in a triangle that are multiples of 30°.

5. Extended Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

6. Criteria for Cyclic Quadrilaterals: It is essential to know certain properties of quadrilaterals that can be inscribed in circles for olympiads, and it can also be useful on the AIME.

   - Opposite angles of a cyclic quadrilateral add up to 180 degrees.
   - Pairs of inscribed angles (such as $\angle ABD$ and $\angle ACD$) are equal.
   - Power of a Point relations hold.

   Proving any of these criteria for any quadrilateral proves that the quadrilateral is cyclic and results in the other properties being true (this can give useful information on many angles).

7. Ptolemy's Theorem (for cyclic quadrilaterals): If the vertices of a cyclic quadrilateral are A, B, C, D (in order!), then

$$(AB)(CD) + (AD)(BC) = (AC)(BD)$$

8. Stewart's Theorem : A man and his dad put a bomb in the sink: $man + dad = bmb + cnc$. This is essentially law of cosines twice (why?), but in a very nice format.

9. Brahmagupta's Formula: For a cyclic quadrilateral with sides of length a, b, c, d, semiperimeter s, and area K

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

   This is a generalization of Heron's formula.

10. Mass Points or Barycentric Coordinates (also called areal coordinates): We will not cover either today due to time, but I suggest you look into these if you have time over the summer. I prefer the latter as it is more general and can apply to olympiads.

With this you have the prerequisite knowledge to solve any AIME geometry problem, up to #15. Of course, this is easier said than done. It is important that you practice as much as possible, and know some general guidelines.

3 Strategies

1. Draw an accurate diagram. This is perhaps the most important strategy out of all presented. On the AIME you have 3 hours, so make sure to bring a ruler and compass and draw full-scale diagrams using the entire page. Multiple colors can also help on complicated diagrams.

2. Don't stare. When starting a geometry problem, it is best to dive right into it. Assign variables and draw lines - don't just hope something will pop out.

3. "Bash" moderately. Most AIME problems will have a nice solution, but this is not always the case. I reccomend you only "bash" if you see a clear path to the solution that will take 20 minutes or less. Coordinates are rarely useful, but there have been a few cases where they have killed a problem. Clues to look at trig are angles that are multiples of 15°. The important thing to remember here is to not bash stupidly.

4. Use all the information in the problem statement. If you have partial progress, look which length or part of the configuration has not been used yet, and then incorporate it.

5. For the somewhat rare 3-d problems, look at relevant cross-sections. For hard problems, you will have to take more than one.

6. If you have parallel lines, look for similar triangles. If you don't have parallel lines, think about adding some.

7. Rarely, the committee will propose a problem that is more general than it needs to be. For example, the problem might be about any parallelogram but there is nothing that prevents it from being a simple square. If the problem is of this type, you may assume a less general but simpler configuration, which can greatly simplify the solution. Just be careful that with this "assuming" you do not contradict the problem statement.

4 Problems

Point values are given after the problem depending on their placement on AIME. The AIME writers are not perfect however, and some later problems are somewhat simple while some earlier problems have proven to be much harder than expected (#4).

1. On square $ABCD$, point E lies on side $\overline{AD}$ and point F lies on side $\overline{BC}$, so that $BE = EF = FD = 30$. Find the area of square $ABCD$. [2]

2. Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base EM, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to EM in $\triangle GEM$. [2]

3. A circle with diameter $\overline{PQ}$ of length 10 is internally tangent at P to a circle of radius 20. Square $ABCD$ is constructed with A and B on the larger circle, $\overline{CD}$ tangent at Q to the smaller circle, and the smaller circle outside $ABCD$. The length of $\overline{AB}$ can be written in the form $m + \sqrt{n}$, where m and n are integers. Find $m + n$. [2]

4. In triangle ABC, $AB = 125, AC = 117$, and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN. [4]

5. One base of a trapezoid is 100 units longer than the other base. The segment that joins the midpoints of the legs divides the trapezoid into two regions whose areas are in the ratio 2 : 3. Find the sum of the lengths of the two bases. [6]

6. Circles C_1 and C_2 are externally tangent, and they are both internally tangent to circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively, and the centers of the three circles are all collinear. A chord of C_3 is also a common external tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$ where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$. [8]

7. A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. The length of the line segment along which the triangle is folded can be written as $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$. [9]

8. Let $ABCDEF$ be a regular hexagon. Let G, H, I, J, K, and L be the midpoints of sides AB, BC, CD, DE, EF, and AF, respectively. The segments AH, BI, CJ, DK, EL, and FG bound a smaller regular hexagon. Let the ratio of the area of the smaller hexagon to the area of $ABCDEF$ be expressed as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$. [9]

9. Let $\triangle PQR$ be a triangle with $\angle P = 75°$ and $\angle Q = 60°$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside $\triangle PQR$ so that side $\overline{AB}$ lies on $\overline{PQ}$, side $\overline{CD}$ lies on $\overline{QR}$, and one of the remaining vertices lies on $\overline{RP}$. There are positive integers a, b, c, and d such that the area of $\triangle PQR$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$. [12]

10. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$. [15]