Polynomials

Epsilon Summer Series

July 9, 2015

1 Finding Roots of Polynomials

What is a polynomial? I'm sure you're all familiar with a quadratic equation.

Q1: What is the quadratic formula? Can you derive it?

A polynomial is of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the a_i are the coefficients and n is the degree. We say that r is a root of P if $P(r) = 0$

Consider the polynomial $P(x) = 5x^2 - 13x + 6$
a) What is its degree?
b) What are its coefficients?
c) What are its roots?

Fact: If r is a root of a polyomial P, then $P(x) = (x - r)Q(x)$ for some polynomial $Q(x)$.
Note that Q must have degree $n - 1$. We can deduce that a polynomial with degree n has at most n roots.

There are a few different methods to find the roots of a polynomial:

1. **Degree 1 and 2:** Linear equations are simple, for degree 2 polynomials use the quadratic formula.
2. **Cubic Formula** If you know how to solve a cubic, good for you. If you don't, that's fine. Use this only as a last resort. Beware pitfalls; you will often get real numbers expressed as a sum of cube roots of complex numbers, and these may or may not simplify.
3. **Rational Root Theorem:** If the coefficients of a polynomial are integers, and there exists a rational root of the form $\frac{p}{q}$ such that $gcd(p, q) = 1$, then p must divide a_0 and q must divide a_n. After finding this root, divide the polynomial by $(qx - p)$
4. **Symmetry and Cleverness:** Sometimes there won't be a rational root. If you want to find a root you might need to use a combination of substitutions and symmetry to factor the polynomial. (Or reconsider and try a different approach).

It might also be useful to look at roots of polynomials + something else. For example, if you know that $P(1) = P(10) = P(100) = 1$, then the polynomial $P(x) - 1$ has roots 1, 10, and 100, so

$$P(x) = Q(x)(x - 1)(x - 10)(x - 100) + 1$$

.

Example: If f is a polynomial of degree 4, such that $f(0) = f(1) = f(2) = f(3) = 1$, and $f(4) = 0$, find $f(5)$

2 Algebraic Manipulations

As a general tip for all algebra problems, sometimes a problem will be more than finding a root or using a formula. In order to solve it, a clever substitution or manipulation of the given equation may be necessary.You should attempt to exploit symmetry in the problem itself, or transform an unfamiliar expression into something you know how to work with.

Example: Determine all r such that $(r^2 + 5r)(r^2 + 5r + 3) = 4$

3 Vieta's Formula

Let P be a polynomial with roots $r_1, r_2, \dots r_n$. Then

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\dots(x - r_n).$$

Expanding the right hand side and comparing coefficients, we deduce that

$$\frac{a_{n-1}}{a_n} = -(r_1 + r_2 + \cdots + r_n)$$

$$\frac{a_{n-2}}{a_n} = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_2r_n + \dots r_{n-1}r_n$$

And so on, until

$$\frac{a_0}{a_n} = (-1)^n r_1 r_2 \dots r_n$$

Basically, the ratio of the ith coefficient to the nth equals $(-1)^i$ times the sum of all possible products of i distinct roots. Most of the time we will only care about the degree 2 and 3 cases, so if $P(x) = ax^2 + bx + c$ with roots r, s, then

$$r + s = -\frac{b}{a}, rs = \frac{c}{a}$$

If $P(x) = ax^3 + bx^2 + cx + d$, with roots r, s, t, then

$$r + s + t = -\frac{b}{a}, rs + rt + st = \frac{c}{a}, rst = -\frac{d}{a}$$

Question: What would the expressions be for a degree 4 polynomial?

3.1 Manipulating Roots

Often times you will be unable to apply vieta's formula directly and instead you will be giving some expression in terms of roots. You should know how to manipulate the symmetric sums in vieta's formula to produce other expressions.

Some common expressions are:

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$$

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) + 3abc$$

Another fact: by reversing the coefficients of a polynomial, you get another polynomial whose coefficients are reciprocals of the original one.

Question: Find $a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$ in terms of the symmetric sums.

4 Problems

1. Determine all ordered pairs a and b such that $a^2 + b^2 = 40$ and $ab = 12$

2. The sum of the zeroes, the product of the zeroes, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Show that all three of these quantities must equal a

3. Find all real solutions to $x^4 + (2 - x)^4 = 34$

4. What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$

5. There is a unique polynomial of the form $P(x) = 7x^7 + c_1x^6 + c_2x^5 + c_3x^4 + c_4x^3 + c_5x^2 + c_6x + c_7$ such that $P(1) = 1$, $P(2) = 2$, ... ,$P(7) = 7$. Find $P(0)$

6. Determine the positive difference between the roots of $x^2 - px + \frac{p^2-1}{4}$

7. Determine all x such that $x^4 + x^3 + x^2 + x + 1 = 29x^2$

8. Let $P(x)$ be a polynomial of degree 1996. If $P(n) = \frac{1}{n}$ for n = 1,2,3,...,1997, compute the value of $P(1998)$

9. Let r, s, and t be the three roots of the equation
$$8x^3 + 1001x + 2008 = 0.$$
Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

10. (Hardest AMC 10 problem ever) Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that
$$P(1) = P(3) = P(5) = P(7) = a, \text{ and}$$
$$P(2) = P(4) = P(6) = P(8) = -a.$$
What is the smallest possible value of a?

11. Determine all roots, real or complex, of the system of simultaneous equations
$$x + y + z = 3$$
$$x^2 + y^2 + z^2 = 3$$
$$x^3 + y^3 + z^3 = 3$$