Geometric Proofs

Epsilon Summer Series Class 6

August 14, 2014

1 Topics

- Equal, supplementary, complementary angles with parallel and perpendicular lines
- Cyclic quadrilaterals: Quadrilateral ABCD is *cyclic* if all its vertices lie on a circle. Then:
  - $\angle ABD = \angle ACD$, $\angle ADB = \angle ACB$, $\angle BDC = \angle BAC$, $\angle CAD = \angle CBD$
  - $\angle ABC + \angle ADC = 180$, $\angle BCD + \angle BAD = 180$.

  Note that these criteria work in the other direction, and can be used to show a quadrilateral is cyclic.
- Let ABC be a triangle, and D be a point such that DA is tangent to the circumcircle of ABC and D is on the same side as B. Then $\angle DAB = \angle ACB$.

  These facts can be used to determine equal angles in a method known as angle-chasing.
- Know SSS, SAS, ASA, AA congruences and similarities!
- Power of a point: Let P be a point and ω be a circle. Let ℓ be an arbitrary line passing through P and intersecting ω at A and B. The product $PA \cdot PB$ is constant. This constant is known as the power of P with respect to ω.

  One convention is that if P is inside ω, then the power is negative, and if P is outside, the power is positive. Then, if ω has center O and radius r, the power of P is $PO^2 - r^2$.
- Radical Axis: Given two circles ω_1, ω_2, what is the locus (the set) of all points that have equal powers with respect to both circles? Using coordinates we can show that this locus, known as the radical axis, is a line. Thus if two circles intersect at X, Y, their radical axis is XY. Given three circles, the three radical axes concur (why?).
- Triangle centers: The perpendicular bisectors (circumcenter), medians (centroid), angle bisectors (incenter), and altitudes (orthocenter) concur. Know the proofs for each. Also be able to find equal angles in each configuration.
- Homothety: a dilation centered at a point.

2 Cool Configurations

1. **The orthocenter and the circumcenter:**

   First, there are a TON of cylic quads and equal angles in the orthocenter and altitudes configuration. Try to find them all.

   The reflections of the orthocenter over the sides and midpoints of sides lie on the circumcircle.

   The orthocenter is the incenter of the orthic triangle.

   The orthocenter (H) and circumcenter(O) are isogonal conjugates. What does that mean? It means $\angle BAO = \angle CAH$, $\angle CBO = \angle ABH$, $\angle ACO = \angle BCH$.

2. **Incenter and Excenters:**

   Let the angle bisector of A intersect the circumcircle at M. Then M is the midpoint of the arc BC.

   The incenter is the orthocenter of the triangle formed by the excenters.

   Let the A-Excenter (E) be the center of the other circle tangent to BC and the rays AB and AC. Note that E must be the interesection of the angle bisector of A with the external angle bisectors of B and C. We then have that $IBEC$ is cyclic with center M.

   If the A-excircle intersects BC at F, and the incircle intersects BC at D, then $BD = CF$.

3. **Nine-point circle and Euler line:**

   The centroid, circumcenter, and orthocenter are concurrent! They lie on the line known as the Euler line. Also, G divides the segment OH in a $1 : 2$ ratio.

   Let AH, BH, CH intersect BC, AC, AB at D, E, F, let the midpoints of the sides be M, N, P, and let the midpoints of AH, BH, CH be X, Y, Z. Then D, E, F, M, N, P, X, Y, Z all lie on a circle! This is known as the nine point circle, and its center is the midpoint of OH.

3 Computational Methods

Aside from angle chasing and similar triangles, problems can be solved with computational methods too! Remember:

- Law of sines/cosines
- Trig (with right angles)
- Area ratios
- Pythagorean theorem
- Coordinates (cartesian, complex numbers, barycentric)

4 Problems

1. Prove the angle bisector theorem.
2. Let ABCD be a cyclic quadrilateral and denote by P its intersection of diagonals. Let circle ω passing through A and B intersect segments PC, PD, at X, Y respectively. Prove that XY is parallel to CD
3. Let circles ω_1, ω_2 intersect at A, B and let P be a point on the segment AB. Line ℓ_1 through P intersects ω_1 at K and L, and line ℓ_2 through P intersects ω_2 at M and N. Prove that points K, L, M, N lie on a circle.
4. Circles ω, ω' intersect at A and B. An arbitrary line through A intersects them for a second time at C, D, respectively. The tangents at C, D to the respective circles intersect at P. Prove that P lies on the circumcircle of BCD.
5. Let L and K be points on the lines AC and BC. Prove that the common hcord of the circles with diameters AK and BL passes through the orhtocenter H of ABC.
6. (JBMO 2010) Let BK and CL be angle bisectors in an acute triangle ABC with incenter I. The perpendicular bisector of LC intersects the line BK at point M. Point N lies on the line CL such that NK is parallel to LM. Prove that $NK = NB$.
7. (Simson Line) Let P be a point on the circumcircle of ABC. Prove that the projections of P onto the sides of ABC are collinear.
8. (2012 USAJMO #1) Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).
9. Let the incircle of triangle ABC intersect BC at D, let the A-excircle intersect BC at E, and let F be the point diametrically opposite D on the incircle. Prove A, F, E are collinear.
10. (2013 IMO #4) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogoously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.