Combinatorics

Epsilon Summer Series

July 23, 2015

1 Introductory Problems

1. How many ways are there to arrange the letters of the word $MISSISSIPPI$?
2. **Block Walking:** How many lattice paths are there from $(0, 0)$ to $(5, 7)$ going only up and right?
3. **Stars and Bars:** How many solutions in nonnegative integers are there to $a + b + c + d = 7$?
4. **PIE:** In a school with 500 students, 276 take Spanish, 54 play baseball, 38 are in the band, 43 play baseball and take Spanish, 20 are in the band and take Spanish, 7 play baseball and are in the band, and 2 do all 3. How many students do none of the activities?
5. **Geometry:** John and Jane are invited to a 90 minute party. Each will arrive at some time during the party, and stay for 30 minutes or until the party ends. What is the probability they will see each other?
6. **Recursion:** How many ways are there to tile a 1 by 6 rectangle with 1 by 1 and 1 by 2 rectangles?
7. **States:** What is the expected number of flips of a coin to get 3 heads in a row?
8. **Constructive Counting:** How many 5 digit numbers have no two adjacent digits equal?
9. **Just Do It:** The numbers 1, 2, 3, 4, 5 are to be arranged in a circle. An arrangement is *bad* if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?

2 Bigger Challenges

1. A pair of six-sided fair dice are labeled so that one die has only even numbers (two each of 2, 4, and 6), and the other die has only odd numbers (two each of 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on top of the two dice is 7?
2. How many positive integers less than 1000 are multiples of 2 and 5 but not both?
3. Consider the set $S = \{1, 2, 3, \dots, 34\}$. How many ways are there to choose (without regard to order) three numbers from S whose sum is divisible by 3?
4. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is the rotated $90°$ clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?
5. Real numbers x, y, and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?
6. In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0 < N < 10$, it will jump to pad $N - 1$ with probability $\frac{N}{10}$ and to pad $N + 1$ with probability $1 - \frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape without being eaten by the snake?

7. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:
(i) Any cube may be the bottom cube in the tower.
(ii) The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.
Let T be the number of different towers that can be constructed. What is the remainder when T is divided by 1000?

8. Let $A = \{1, 2, 3, 4\}$, and f and g be randomly chosen (not necessarily distinct) functions from A to A. What is the probability that the range of f and the range of g are disjoint?

9. You have a robot. Each morning the robot performs one of four actions, each with probability 1/4:
   - Nothing.
   - Self-destruct.
   - Create one clone.
   - Create two clones.

   Compute the probability that you eventually have no robots.

10. Ten chairs are arranged in a circle. Find the number of subsets of this set of chairs that contain at least three adjacent chairs.