Algebra Part II: Polynomials, Complex Numbers, and Trigonometry

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1 Polynomials

1.1 Finding Roots of Polynomials

A polynomial is of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the a_i are the coefficients and n is the degree. We say that r is a root of P if $P(r) = 0$
Fact: If r is a root of a polyomial P, then $P(x) = (x - r)Q(x)$ for some polynomial $Q(x)$.
Note that Q must have degree $n - 1$. We can deduce that a polynomial with degree n has at most n roots.

There are a few different methods to find the roots of a polynomial:

1. **Degree 1 and 2:** Linear equations are simple, for degree 2 polynomials use the quadratic formula.
2. **Rational Root Theorem:** If the coefficients of a polynomial are integers, and there exists a rational root of the form $\frac{p}{q}$ such that $gcd(p, q) = 1$, then p must divide a_0 and q must divide a_n. After finding this root, divide the polynomial by $(qx - p)$
3. **Symmetry and Cleverness:** Sometimes there won't be a rational root. If you want to find a root you might need to use a combination of substitutions and symmetry to factor the polynomial. (Or reconsider and try a different approach).

It might also be useful to look at roots of polynomials + something else. For example, if you know that $P(1) = P(10) = P(100) = 1$, then the polynomial $P(x) - 1$ has roots 1, 10, and 100, so

$$P(x) = Q(x)(x - 1)(x - 10)(x - 100) + 1$$

.

1.2 Vieta's Formula

Let P be a polynomial with roots $r_1, r_2, \ldots r_n$. Then

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\ldots(x - r_n).$$

Expanding the right hand side and comparing coefficients, we deduce that

$$\frac{a_{n-1}}{a_n} = -(r_1 + r_2 + \cdots + r_n)$$

$$\frac{a_{n-2}}{a_n} = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_2r_n + \ldots r_{n-1}r_n$$

And so on, until

$$\frac{a_0}{a_n} = (-1)^n r_1 r_2 \ldots r_n$$

Basically, the ratio of the ith coefficient to the nth equals $(-1)^i$ times the sum of all possible products of i distinct roots. Most of the time we will only care about the degree 2 and 3 cases, so if $P(x) = ax^2 + bx + c$ with roots r, s, then

$$r + s = -\frac{b}{a}, rs = \frac{c}{a}$$

If $P(x) = ax^3 + bx^2 + cx + d$, with roots r, s, t, then

$$r + s + t = -\frac{b}{a}, rs + rt + st = \frac{c}{a}, rst = -\frac{d}{a}$$

Question: What would the expressions be for a degree 4 polynomial?

1.3 Manipulating Roots

Often times you will be unable to apply vieta's formula directly and instead you will be giving some expression in terms of roots. You should know how to manipulate the symmetric sums in vieta's formula to produce other expressions.

Some common expressions are:

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$$

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) + 3abc$$

Another fact: by reversing the coefficients of a polynomial, you get another polynomial whose coefficients are reciprocals of the original one.

Question: Find $a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$ in terms of the symmetric sums.

2 Complex Numbers

Consider the polynomial $x^2 + 1$. What are its roots? Since the square of any real number is nonnegative, its roots are clearly not real. We define the number "i" to be a root of this equation, or that $i = \sqrt{-1}$. A complex number is of the form $z = a + bi$, where a and b are real numbers and $i^2 = -1$. If $a = 0$, we call this number imaginary. The conjugate of a complex number, $\bar{z}$, equals $a - bi$.
Q1: What is $|5 + 12i|$?
Q2: What is $z\bar{z}$?
Q3: Show that $\overline{w + z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$
.

This allows us to look at roots of a polynomial from a completely different perspective. Earlier we claimed that a polynomials of degree d has at most d roots. It turns out that a polynomial of degree d has *exactly* d complex roots (including multiplicity). Another fact is that if P is a poynomial with real coefficients, and $z = a + bi$ is a complex root of P, then the conjugate of z, $a - bi$, must also be a root.

2.1 The Complex Plane

Oftentimes it is useful to look at the graphical representations of complex numbers. We do this with the complex plane - see Figure 1 below. We can interpret complex numbers graphically in a 2-d cartesian coordinate plane. To do this let the x-axis be the "real axis" and the y-axis be the "imaginary axis", and let the complex number $a + bi$ correspond to the point (a, b). $|z|$, pronounced as "magnitude z", is equal to $\sqrt{a^2 + b^2}$ for $z = a + bi$. (It is therefore like absolute value for real numbers, as it represents the distance from z to the origin).

Question 1: For complex numbers, w and z, what does $|w - z|$ represent?
Question 2: Describe the graph of $|z - 3| = 4$.

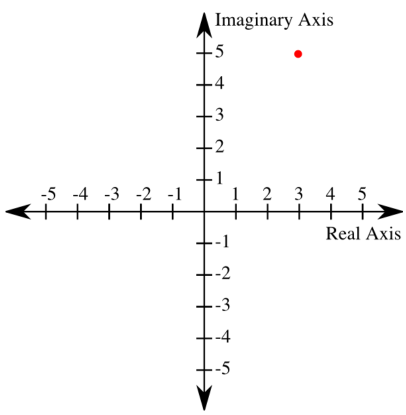

Figure 1: This point would represent the complex number $3 + 5i$.

2.2 Polar Form and de Moivre's Theorem

By looking at the complex plane, we see that it is possible to represent a complex number in polar form, or by using the magnitude and angle it makes with the positive real axis as the parameter. Converting polar coordinates to rectangular, a complex number with magnitude r and argument θ (angle with real axis) equals $r\cos(\theta) + ir\sin(\theta)$.

Q1: Let $w = 3 + i\sqrt{3}$. Write w in polar form.
Q2: Let $w = r(\cos\alpha + i\sin\alpha)$. and $z = s(\cos\beta + i\sin\beta)$. What is wz in polar form? What does this tell us?

Q2 implies that complex numbers have an exponential nature. In fact, complex numbers can be written in exponential form! We define

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Thus each complex number can also be represented in the form $re^{i\theta}$ for some real r, θ (theta must be in radians). This allows us to interpret multiplying complex numbers in a very interesting way - when multiplying two complex numbers simply add their arguments and multiply their magnitudes. Another theorem, known as $DeMoivre's\ Theorem$, states that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

This is a direct consequence of the formula shown above.

2.3 Roots of Unity

Consider the roots of the polynomial $x^n - 1 = 0$ for some positive integer n. Obviously 1 is a root, but let's look at the complex roots. Let z be a complex root, we can write it as $z = re^{i\theta}$ for some r, θ. Then $r^n e^{in\theta} = 1$. Comparing magnitudes, we see that r must equal 1, so we get that $e^{in\theta} = 1$. By our formula we get that

$$\cos(n\theta) + i\sin(n\theta) = 1$$

which means that $n\theta$ must be a multiple of 2π. Thus $\theta = \frac{2k\pi}{n}$ for some k. Since k can be any integer ranging from 0 to n-1, we see that

$$1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{2(n-1)\pi i}{n}}$$

are precisely the n roots of the original polynomial. These are called the nth roots of unity. In fact, if we let $\omega = e^{\frac{2\pi i}{n}}$, we get that

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1}).$$

Note that if a complex number is a root of unity, its conjugate is (why?) and that the roots of unity lie on the unit circle and form a regular n-gon in the complex plane.

3 Trigonometry

There is almost always a trig problem on each AIME. The basics of trig is necessary for almost all problems, in both geometry and algebra. In the simplest case of a right triangle, remember SOH-CAH-TOA, where $\sin x = \frac{opposite}{hypoteneuse}$, $\cos x = \frac{adjacent}{hypoteneuse}$, $\tan x = \frac{opposite}{adjacent}$.

3.1 Trig Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin(x) = \cos(\frac{\pi}{2} - x)$$

$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-tan(a)tan(b)}$$
$$\tan(a-b) = \frac{\tan(a)-\tan(b)}{1+tan(a)tan(b)}$$

It is definitely important to remember the addition formulas, but one interesting way to derive the last 4 is by using complex numbers:

$$\cos(a+b) + i\sin(a+b) = e^{i(a+b)} = e^{ia}e^{ib} = (\cos(a) + i\sin(a))(\cos(b) + i\sin(b))$$

Now just expand the right hand side and compare the real and imaginary parts.

Sum to Product:

$$\cos(a)\cos(b) = 1/2(\cos(a+b) + \cos(a-b))$$
$$\sin(a)\sin(b) = 1/2(\cos(a-b) - \cos(a+b))$$
$$\sin(a)\cos(b) = 1/2(\sin(a+b) + \sin(a-b))$$
$$\cos(a)\sin(b) = 1/2(\sin(a+b) - \sin(a-b))$$

It's ok if you don't memorize the product to sum identities (it might help with speed), but its important to know how they are derived from the sum and difference formulas.

Q: Compute $\cos(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

One way is to use the angle addition formula. However, it's also possible to use DeMoivre's Theorem. Consider

$$\cos(3\theta) + i\sin(3\theta) = (\cos(\theta) + i\sin(\theta))^3$$
$$= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Comparing real and imaginary parts, we get that

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$
$$\sin(3\theta) = \cos^2(\theta)\sin(\theta) - \sin^3(\theta)$$

Q2: Find $\cos(5\theta)$

3.2 Techniques for Solving Trig Problems

Similar to most other algebra problems, problems involving trig can be solved with substituting, exploiting symmetry, telescoping, and using the identities in the previous section. Don't be intimidated by trig! Just try to play around using the identities you know, and solve it the same as you would any other algebra problem.

For harder problems complex numbers may be helpful, consider looking at the real and imaginary parts or use exponential form/DeMoivre's Theorem.

4 Problems

1. The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

2. Suppose the funtion $x^3 + x - 1$ has roots a, b and c. Find the value of $\frac{a^3}{1-a} + \frac{b^3}{1-b} + \frac{c^3}{1-c}$.

3. Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a, \text{ and}$$

$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

4. The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin\left(\frac{m\pi}{n}\right)$ for relatively prime positive integers m and n with $m < n$. Find n.

5. Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x)R(x)$?

6. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

7. Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2a)\sin(na)]$$

is an integer.

8. For how many positive integers n less than or equal to 1000 is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?

9. The solutions of the equation $z^4 + 4z^3i - 6z^2 - 4zi - i = 0$ are the vertices of a convex polygon in the complex plane. What is the area of the polygon?

10. There are nonzero integers a, b, r, and s such that the complex number $r+si$ is a zero of the polynomial $P(x) = x^3 - ax^2 + bx - 65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}$'s for all possible combinations of a and b.

11. (Roots of Unity Filter) Determine

$$\binom{3k}{0} + \binom{3k}{3} + \binom{3k}{6} + \cdots + \binom{3k}{3k-3} + \binom{3k}{3k}$$

12. (Hard) Compute $\arctan(\tan 65 - 2\tan 40)$.