# Algebra I

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## 1 Sequences and Series

A sequence is essentially defined as any list of numbers. A series is simply the sum of the terms of a particular sequence (for example, $\sum_{n=0}^{\infty} \frac{1}{n}$).
Arithmetic sequences are perhaps the most basic - they are defined by a starting term and a common difference, so they can be represented as

$$a, a+d, a+2d, \dots a+(n-1)d$$

The sum of the first $n$ terms is $na + \frac{n(n-1)}{2}d$, by a simple argument.

Geometric sequences have a common ratio (as opposed to a common difference), and can be represented by

$$a, ar, ar^2, \dots ar^{n-1}$$

The sum of the first $n$ terms is $a\frac{r^n-1}{r-1}$

Proof: Let

$$S = a + ar + ar^2 + \cdots + ar^{n-1}$$

Then

$$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

So

$$Sr - S = ar^n - a$$
$$\rightarrow S = a\frac{r^n - 1}{r - 1}$$

If $|r| < 1$, then the sum of an infinite geometric series will converge, and we have the well-known formula

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

Arithmetico-Geometric Series include components of both arithmetic and geometric series. To solve them, use the same techniques used in the proof of the geometric series formula.

*Example:* Find

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \dots$$

*Solution:* Let this sum equal S. Then

$$2S = 1 + \frac{2}{2} + \frac{3}{4} + \cdots + \frac{n+1}{2^n} + \dots$$

Subtracting,

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$$

Another extremely important technique when working with series is telescoping, where if we want to find $\sum_{i=1}^{n} g(i)$, we write $g(x) = f(x+1) - f(x)$ for some $f$, and see that all the terms cancel out except for the first and last.

*Example:* Find $\sum_{i=1}^{100} \frac{1}{i(i+1)}$

*Solution:* Since $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$, we can say $\sum_{i=1}^{100} \frac{1}{i(i+1)} = \sum_{i=1}^{100} \frac{1}{i} - \frac{1}{i+1} = 1 - \frac{1}{101} = \frac{100}{101}$ ( all the terms except for those two will cancel out).

# 2 Algebraic Tricks

In this section we'll look at tricks and strategies that will help solve problems that ask for the solution of a set of equations. Most often, such problems require creativity and problem-solving experience.

In problems with multiple equations, you may find that adding them up cancels several annoying terms. Try to use this to your advantage.You may also find that adding or subtracting quantities from both sides can significantly simplify an 'ugly' expression. Always be on the lookout for these manipulations.

## 2.1 Common Factorizations

1. Difference of Squares:
$$x^2 - y^2 = (x+y)(x-y)$$

2. Simon's Favorite Factoring Trick:
$$xy + ax + by = (x+b)(y+a) - ab$$

3. Sum and Difference of Cubes:
$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

4. Sophie Germain Identity (learn this!):
$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$$

5. Sum and Difference of Powers:
$$x^n - a^n = (x^{n-1} + ax^{n-2} + a^2x^{n-3} ... + a^{n-1})(x-a)$$
When n is odd:
$$x^n + a^n = (x^{n-1} - ax^{n-2} + a^2x^{n-3} ... + a^{n-1})(x+a)$$

6. Interesting factorization:
$$a^3 + b^3 + c^3 = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) + 3abc$$
A neat consequence of Facotrization #6 is that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

## 2.2 Substitutions

Substitutions are a simple yet powerful technique. However, beware that substitutions do not always make a problem easier. If a substitution makes any part of the equation(s) hard to simplify, it may not be the substitution you are looking for. Common approaches to try include:

- If an expression appears multiple times, it may be a good idea to replace it with a variable.
- Symmetry: Try to exploit the symmetry in a problem, as this will significantly help in your computations. For example, if you see many x+2 and x+10 expressions in a problem, you may consider the substitution y = x+6.
- If you know that $n$ is an odd or even integer, replace it $2m+1$ or $2m$. Usually in diophantine equations (equations where you want to find integer solutions), it is helpful to replace integers by other integers using what you know about them.

### 2.3 Properties of Squares

At a basic level, this means completing the square. Look out for common terms after completing the square when you encounter a quadratic, as you may be able to cancel out certain terms to simplify the equation(s).

A simple yet important property of squares is the fact that $x^2 \geq 0$ for all real x. If you are able to simplify an equation or set of equations into a bunch of squares, you may be able to eliminate a range of numbers and be sure you have found all solutions to the equation(s).

## 3 Logarithms

Logarithms do not appear on on the AMC 10, but they are on the AMC 12 and AIME. Problems involving logs can be solved with the strategies mentioned above along with a few identities that are important to know:

$$a^x = b \to \log_a b = x$$
$$\log a + \log b = \log ab$$
$$\log a - \log b = \log(a/b)$$
$$\log_a b = \frac{\log_c b}{\log_c a}$$
$$a^{\log_a b} = b$$
$$\log_a b = \frac{1}{\log_b a}$$

Keep in mind that the domain of a log function is $x > 0$.

# 4 Problems

1. The domain of the function $f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of length $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?

2. There exist unique positive integers $x$ and $y$ that satisfy the equation $x^2 + 84x + 2008 = y^2$. Find $x + y$

3. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where $m$ and $n$ are relatively prime integers. Find $m + n$

4. Determine the number of ordered pairs $(a, b)$ of integers such that $\log_a b + 6 \log_b a = 5$, $2 \leq a \leq 2005$, and $2 \leq b \leq 2005$

5. Find the positive solution to
$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0$$

6. Positive numbers $x$, $y$, and $z$ satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$

7. Find
$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{5}{81} + \cdots + \frac{F_n}{3^n} + \ldots$$
where $F_n$ represents the nth Fibonacci number.

8. Compute
$$\frac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)}$$

9. Let $m$ be the largest real solution to the equation
$$\frac{3}{x - 3} + \frac{5}{x - 5} + \frac{17}{x - 17} + \frac{19}{x - 19} = x^2 - 11x - 4.$$
There are positive integers $a, b, c$ such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

10. A sequence is defined as follows $a_1 = a_2 = a_3 = 1$, and, for all positive integers $n$, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$. Given that $a_{28} = 6090307$, $a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when $\displaystyle\sum_{k=1}^{28} a_k$ is divided by 1000.

11. Let $(x, y)$ be an intersection of the equations $y = 4x^2 - 28x + 41$ and $x^2 + 25y^2 - 7x + 100y + \frac{349}{4} = 0$. Find the sum of all possible values of $x$.